Sofia University “St. Kliment Ohridski”
Faculty of Mathematics and Informatics

Simultaneous approximation by the Bernstein operator

Borislav Radkov Draganov

Abstract
of
a Dissertation for Doctor of Sciences Degree in Mathematics

Sofia • 2023

Declaration of Authorship

The author hereby declares that the dissertation contains original results he obtained. All other researchers' results used are acknowledged as references.

Borislav Radkov Draganov

S. N. Bernstein introduced in 1912 an approximation operator, which is now named after him, in order to give a simple proof of Weierstrass's theorem that every continuous function on a finite closed interval can be uniformly approximated by algebraic polynomials [4].

The Bernstein operators or polynomials are defined for $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}_+$ by

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}. \tag{1}$$

We have

$$\lim_{n\to\infty} B_n f(x) = f(x) \quad \text{uniformly on} \quad [0,1],$$

that is

$$\lim_{n\to\infty} \|B_n f - f\| = 0, \quad f \in C[0,1],$$

where $\| \circ \|$ stands for the (essential) supremum norm on the interval $[0,1]$.

Moreover, clearly

$$\|B_n f\| \le \|f\|, \quad f \in C[0,1],\ n \in \mathbb{N}_+.$$

Thus $\{B_n\}_{n=1}^{\infty}$ is a strong approximation process on $C[0,1]$ (see [8, Definition 12.0.1]).

Various estimates of the supremum norm of the error $B_n f(x) - f(x)$ were established. Some of the earliest ones were stated in terms of the so-called moduli of smoothness (or continuity). For example, Popoviciu [48] (or see [43, Theorem 1.6.1]) showed that

$$\|B_n f - f\| \le \frac{5}{4}\, \omega_1(f, n^{-1/2}).$$

Above $\omega_1(f,t)$ is the modulus of continuity of f, defined by

$$\omega_1(f,t) := \sup_{|x-y|\le t} |f(x) - f(y)|. \tag{2}$$

Since $B_n f$ interpolates f at the ends of the interval, one can expect that it approximates the function better in their neighbourhood. This is indeed so. The following estimate holds for $f \in AC_{loc}^1(0,1)$ such that $\varphi^2 f'' \in L_\infty[0,1]$, where $\varphi(x) := \sqrt{x(1-x)}$ (see e.g. [12, Chapter 10, § 7] or [15, Chapter 9])

$$\|B_n f - f\| \le \frac{c}{n}\, \|\varphi^2 f''\|, \quad n \in \mathbb{N}_+. \tag{3}$$

Here and henceforward c denotes absolute constants.

This estimate can be further generalized for any $f \in C[0,1]$ and $n \in \mathbb{N}_+$ in the form

$$\|B_n f - f\| \le c\,\omega_\varphi^2(f, n^{-1/2}), \tag{4}$$

where $\omega_\varphi^2(f,t)$ is the Ditzian-Totik modulus of smoothness of second order with varying step controlled by the weight $\varphi(x)$ in the sup-norm on $[0,1]$. It is defined by

$$\begin{aligned} &\omega_\varphi^2(f,t) \\ &\quad := \sup_{0<h\le t} \sup_{x\pm h\varphi(x)\in[0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|, \quad t>0. \end{aligned} \tag{5}$$

Adell and G. Sangüesa [1] proved that (4) holds with $c = 4$. Later on Gavrea, Gonska, Păltănea and Tachev [27] improved this estimate to $c = 3$, then Păltănea [47, p. 96] to $c = 5/2$ (or see [7, p. 183]).

It turns out that (3) and (4) cannot be improved. The converse to (4) is also valid—there holds (see [40] and [53])

$$\|B_n f - f\| \ge c\,\omega_\varphi^2(f, n^{-1/2}), \quad n \ge n_0, \tag{6}$$

where $n_0 \in \mathbb{N}_+$ is independent of f. Earlier, Ditzian and Ivanov [14, Theorem 8.1] obtained a similar two-term converse inequality.

The last estimate implies that $B_n f$ cannot approximate f in the supremum norm on $[0,1]$ with a rate faster than $1/n$ unless $B_n f$ preserves f, that is, f is an algebraic polynomial of degree at most 1. This is known as saturation of an approximation process (see [8, Definition 12.0.2], or [12, p. 336]). Thus the sequence of approximating operators $\{B_n\}_{n=1}^\infty$ is saturated, as its saturation rate is n^{-1}. It was first observed by Voronovskaya [54] (or see e.g. [12, Chapter 10, Theorem 3.1]). She proved that if $f \in C^2[0,1]$, then

$$\lim_{n\to\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2}\, f''(x) \tag{7}$$

uniformly on $[0,1]$.

The Bernstein polynomial possesses another property. As it was established by Chlodowsky [11], Wigert [55] and Lorentz [42] (see e.g. [12, Chapter 10, Theorem 2.1], or [7, p. 232]), it not only approximates the function, but also its derivatives. More precisely, we have

$$\lim_{n\to\infty} (B_n f)^{(s)}(x) = f^{(s)}(x) \quad \text{uniformly on} \quad [0,1], \tag{8}$$

provided that $f \in C^s[0,1]$. That phenomenon is referred to as simultaneous approximation.

The main subject of the dissertation is to present estimates of the rate of this approximation. We prove both direct estimates and matching one- or two-term converse estimates, which show that the direct estimates are sharp. The estimates are established in the essential norm on $[0,1]$ with Jacobi weights. As a further application of those results we characterize the rate of the simultaneous approximation of the iterated Boolean sums of B_n and of two modifications of B_n, which are polynomials with integer coefficients. Finally, we investigate the rate of convergence in Voronovskaya's theorem (7).

Weighted simultaneous approximation by the Bernstein operator

Voronovskaya's result (7) shows that the differential operator which describes the rate of approximation of B_n (up to a constant multiple) is $Df(x) := \varphi^2(x)f''(x)$ with $\varphi(x) := \sqrt{x(1-x)}$. A quantitative description of this rate follows from (4)-(6):

$$\|B_n f - f\| \sim \omega_\varphi^2(f, n^{-1/2}), \quad n \geq n_0, \tag{9}$$

with some $n_0 \in \mathbb{N}_+$, which is independent of $f \in C[0,1]$. We say that $\Phi(f,t)$ and $\Psi(f,t)$ are equivalent and write $\Phi(f,t) \sim \Psi(f,t)$ if there exists a positive constant c such that $c^{-1}\Phi(f,t) \leq \Psi(f,t) \leq c\,\Phi(f,t)$ for all f and t under consideration.

As we indicated earlier in (8), the derivatives of the Bernstein polynomial of a smooth function approximate the corresponding derivatives of the function. López-Moreno, Martínez-Moreno and Muñoz-Delgado [41] and Floater [26] extended (7), showing that for $f \in C^{s+2}[0,1]$ we have

$$\lim_{n\to\infty} n\left((B_n f(x))^{(s)} - f^{(s)}(x)\right) = \frac{1}{2}(Df(x))^{(s)} \quad \text{uniformly on } [0,1]. \tag{10}$$

Hence the differential operator that describes the simultaneous approximation by B_n is $(d/dx)^s D$. Results about the rate of convergence in (10) were established in [29, 30, 32].

The first quantitative result for the simultaneous approximation by means of B_n was given by Popoviciu [49] (or see [7, p. 232]). It states

$$\|(B_n f)^{(s)} - f^{(s)}\| \le \frac{3+2\sqrt{s}}{2}\,\omega_1\left(f^{(s)}, \frac{1}{\sqrt{n-s}}\right) + \frac{s(s-1)}{2n}\,\|f^{(s)}\|, \quad n > s.$$

Numerous improvements of this estimate have been established since then (see [7, Section 4.6]).

To the best of my knowledge, all but one estimate established previously (see Remark 3.6 below) use the classical fixed-step modulus of smoothness of first and second order. The estimates we prove use the Ditzian-Totik modulus and take into account that the approximation is better near the ends of the interval, besides we consider approximation generally in weighted spaces. Moreover, we also establish matching converse inequalities, which show that the direct estimates are sharp. A point-wise direct inequality, which demonstrates that the approximation improves near the ends of the interval was established by Jiang [36] (or see [7, p. 237]), who proved for the first derivative that

$$|(B_n f(x))' - f'(x)| \le \frac{13}{4}\,\omega_2\left(f', \frac{2\varphi(x)}{\sqrt{n-1}}\right) + \omega_1(f', n^{-1}).$$

We consider simultaneous approximation by B_n with the Jacobi weights:

$$w(x) := w(\gamma_0, \gamma_1; x) := x^{\gamma_0}(1-x)^{\gamma_1}, \quad x \in [0,1], \tag{11}$$

where $\gamma_0, \gamma_1 \ge 0$.

To characterize the rate of the simultaneous approximation by B_n, we use the K-functional

$$K_s^D(f,t)_w := \inf_{g \in C^{s+2}[0,1]} \left\{ \|w(f - g^{(s)})\| + t\|w(Dg)^{(s)}\| \right\}.$$

We establish the following direct estimate of the rate of the weighted simultaneous approximation by the Bernstein operator.

Theorem 3.3. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (11) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(B_n f - f)^{(s)}\| \le c\, K_s^D(f^{(s)}, n^{-1})_w.$$

The value of the constant c is independent of f and n.

This estimate can be simplified. The K-functional $K_s^D(f,t)_w$ can be characterized by simpler ones. Let

$$K_m(f,t)_w := \inf_{g\in AC_{loc}^{m-1}(0,1)} \left\{\|w(f-g)\| + t\|wg^{(m)}\|\right\}$$

and

$$K_{m,\varphi}(f,t)_w := \inf_{g\in AC_{loc}^{m-1}(0,1)} \left\{\|w(f-g)\| + t\|w\varphi^m g^{(m)}\|\right\}, \tag{12}$$

where $\varphi(x) := \sqrt{x(1-x)}$. For the unweighted case $w = 1$ we set

$$K_m(f,t) := K_m(f,t)_1$$

and

$$K_{m,\varphi}(f,t) := K_{m,\varphi}(f,t)_1.$$

As we show in Theorems 4.4 and 4.5, if $0 < \gamma_0, \gamma_1 < s$, then for all $wf \in L_\infty[0,1]$ and $0 < t \le 1$ there holds

$$K_s^D(f,t)_w \sim \begin{cases} K_{2,\varphi}(f,t)_w + K_1(f,t)_w, & s = 1, \\ K_{2,\varphi}(f,t)_w + t\,\|wf\|, & s \ge 2, \end{cases} \tag{13}$$

whereas in the case $w = 1$ we have:

$$K_s^D(f,t)_1 \sim \begin{cases} K_{2,\varphi}(f,t) + K_1(f,t), & s = 1, \\ K_{2,\varphi}(f,t) + K_1(f,t) + t\,\|f\|, & s \ge 2, \end{cases} \tag{14}$$

for all $f \in C[0,1]$ and $0 < t \le 1$. The characterization of $K_s^D(f,t)_w$ in the case when one of the γs is 0 and the other is positive is a "mixture" of (13) and (14). The assertion in (13) in the case $s = 1$ actually holds for all $0 \le \gamma_0, \gamma_1 < 1$.

Each of the K-functionals $K_1(f,t)_w$ and $K_{2,\varphi}(f,t^2)_w$ is equivalent to a modulus of smoothness. The latter are function characteristics, which are more directly related to the approximated function than the K-functionals, but are equivalent to them. We have already mentioned two of them—(2) and (5). To extend their definition, we first introduce the difference operator.

The forward difference of $f:[0,1]\to\mathbb{R}$ with step $h>0$ of order $m\in\mathbb{N}_+$ is given by

$$\overrightarrow{\Delta}_h^m f(x) = \begin{cases} \sum_{i=0}^{m}(-1)^i\binom{m}{i} f(x+(m-i)h), & x\in[0,1-mh], \\ 0, & x\in(1-mh,1]. \end{cases}$$

Similarly, the backward difference is given by

$$\overleftarrow{\Delta}_h^m f(x) = \begin{cases} \sum_{i=0}^{m}(-1)^i\binom{m}{i} f(x-ih), & x\in[mh,1], \\ 0, & x\in[0,mh). \end{cases}$$

We also make use of the symmetric difference, which is defined on $[0,1]$ by

$$\bar{\Delta}_h^m f(x) = \begin{cases} \sum_{i=0}^{m}(-1)^i\binom{m}{i} f\left(x+\left(\frac{m}{2}-i\right)h\right), & x\in\left[\frac{mh}{2},1-\frac{mh}{2}\right], \\ 0, & \text{otherwise}. \end{cases}$$

The classical unweighted fixed-step modulus of smoothness of order m of $f\in L_\infty[0,1]$ is then defined for $t>0$ by

$$\omega_m(f,t) := \sup_{0<h\le t}\|\overrightarrow{\Delta}_h^m f\|.$$

The weighted modulus of smoothness $\omega_m(f,t)_w$ is defined by

$$\omega_m(f,t)_w := \sup_{0<h\le t}\|w\overrightarrow{\Delta}_h^m f\|_{[0,3/4]} + \sup_{0<h\le t}\|w\overleftarrow{\Delta}_h^m f\|_{[1/4,1]}.$$

Here $\|\circ\|_J$ denotes the essential supremum norm on $J\subset\mathbb{R}$.

In the case $w=1$ we can use the modulus of smoothness $\omega_m(f,t)$—we set

$$\omega_m(f,t)_1 := \omega_m(f,t).$$

One generalization of the classical moduli, which is equivalent to the K-functional $K_{m,\varphi}(f,t^m)$, was introduced by Ditzian and Totik [15, (2.1.2)]. In the unweighted case $w=1$ it is given by

$$\omega_\varphi^m(f,t) := \sup_{0<h\le t}\|\bar{\Delta}_{h\varphi}^m f\|.$$

The generalization of that modulus of smoothness to the weighted case is more complicated. For $\gamma_0, \gamma_1 > 0$ it is defined by (see [15, Appendix B])

$$\omega_\varphi^m(f,t)_w := \sup_{0<h\le t} \|w\bar{\Delta}_{h\varphi}^m f\|_{[m^2t^2,1-m^2t^2]} + \sup_{0<h\le m^2t^2} \|w\overrightarrow{\Delta}_h^m f\|_{[0,12m^2t^2]} + \sup_{0<h\le m^2t^2} \|w\overleftarrow{\Delta}_h^m f\|_{[1-12m^2t^2,1]}, \tag{15}$$

where $0 < t \le 1/(m\sqrt{2})$.

We set

$$\omega_\varphi^m(f,t)_1 := \omega_\varphi^m(f,t).$$

We have (see [38], [15, Chapters 2 and 6], or [12, Chapter 6])

$$K_m(f,t^m)_w \sim \omega_m(f,t)_w, \quad 0 < t \le 1, \tag{16}$$

and

$$K_{m,\varphi}(f,t^m)_w \sim \omega_\varphi^m(f,t)_w, \quad 0 < t \le t_0, \tag{17}$$

with some $t_0 > 0$, which is independent of f.

By virtue of the last relations, Theorem 3.3 yields the following Jackson-type estimates.

Theorem 3.5. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0,\gamma_1)$ be given by* (11)*. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(B_nf-f)^{(s)}\| \le c \begin{cases} \omega_\varphi^2(f',n^{-1/2})_w + \omega_1(f',n^{-1})_w, & s=1,\ 0\le\gamma_0,\gamma_1<1, \\ \omega_\varphi^2(f^{(s)},n^{-1/2}) + \omega_1(f^{(s)},n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\|, & s\ge 2,\ \gamma_0=\gamma_1=0, \\ \omega_\varphi^2(f^{(s)},n^{-1/2})_w + \dfrac{1}{n}\,\|wf^{(s)}\|, & s\ge 2,\ 0<\gamma_0,\gamma_1<s. \end{cases}$$

The value of the constant c is independent of f and n.

Although the equivalence between $K_{2,\varphi}(F,t^2)$ and $\omega_\varphi^2(F,t)$ was established for $t > 0$ small enough, the direct estimates above are verified for all $n \in \mathbb{N}_+$. In addition, we show that the range of γ_0 and γ_1 in Theorems 3.3

and 3.5 is the broadest possible, which allows direct estimates under natural assumptions on the functions.

Remark 3.6. Jiang and Xie [37] (or see [7, Theorem 4.57]) proved a pointwise direct estimate, which implies the estimate in Theorem 3.5 in the case $s \geq 2,\ \gamma_0 = \gamma_1 = 0$.

The direct estimates stated above are sharp—the following strong converse estimate holds.

Theorem 3.8. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (11) as $0 \leq \gamma_0, \gamma_1 < s$. Then there exists $R \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $k, n \in \mathbb{N}_+$ with $k \geq Rn$ there holds*

$$K_s^D(f^{(s)}, n^{-1})_w \leq c\,\frac{k}{n}\left(\|w(B_nf - f)^{(s)}\| + \|w(B_kf - f)^{(s)}\|\right).$$

In particular,

$$K_s^D(f^{(s)}, n^{-1})_w \leq c\left(\|w(B_nf - f)^{(s)}\| + \|w(B_{Rn}f - f)^{(s)}\|\right).$$

The value of the constant c is independent of f, n and k.

We have stated Theorems 3.3, 3.5 and 3.8 under minimal assumptions on f. However, we have an approximation if and only if $\lim_{t\to 0}\omega_\varphi^2(f^{(s)}, t)_w = 0$ and, in addition, $\lim_{t\to 0}\omega_1(f^{(s)}, t)_w = 0$ in the cases $s = 1,\ 0 \leq \gamma_0, \gamma_1 < 1$ or $s \geq 2,\ \gamma_0 = \gamma_1 = 0$. In the case $w = 1$, we have $\lim_{t\to 0}\omega_1(g,t) = 0$ if and only if $g \in C[0,1]$; similarly $\lim_{t\to 0}\omega_\varphi^2(g,t) = 0$ if and only if $g \in C[0,1]$ (considering two functions which are equal a.e. with regard to the Lebesgue measure as identical); see [15, p. 37]. If $\gamma_0 > 0$, then we must have that $g(x)$ is continuous on $(0,1)$ and $\lim_{x\to 0} x^{\gamma_0} g(x) = 0$; if $\gamma_1 > 0$, then we must have $\lim_{x\to 1}(1-x)^{\gamma_1} g(x) = 0$ (see e.g. [25, p. 94]).

To prove Theorem 3.3 we apply a standard method based on boundedness and Jackson-type estimates for the approximation operator (see e.g. [14, Theorem 3.4]), and to prove Theorem 3.8 we use the method developed by Ditzian and Ivanov [14, Theorem 3.2], which is very effective for such problems. It, too, is based on several estimates that concern the boundedness of the operator and its rate of approximation for smooth functions in various aspects. Let us state these inequalities briefly. In all of them c denotes a constant, whose value is independent of f and n.

The first basic estimate concerns the boundedness of the weighted L_∞-norm of $(B_nf)^{(s)}$.

Proposition 3.14. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (11) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(B_n f)^{(s)}\| \le c\,\|wf^{(s)}\|.$$

The next two are Jackson- and Voronovskaya-type inequalities, respectively.

Proposition 3.17. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (11). Set $s' := \max\{2, s\}$. If $0 < \gamma_0, \gamma_1 \le s$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s+1}(0,1)$ and $wf^{(s')}, w\varphi^2 f^{(s+2)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(B_n f - f)^{(s)}\| \le \frac{c}{n}\left(\|wf^{(s')}\| + \|w\varphi^2 f^{(s+2)}\|\right).$$

If $\gamma_0\gamma_1 = 0$ and still $0 \le \gamma_0, \gamma_1 < s$, then

$$\|w(B_n f - f)^{(s)}\| \le \frac{c}{n}\left(\|wf^{(s')}\| + \|wf^{(s+1)}\| + \|w\varphi^2 f^{(s+2)}\|\right),$$

provided that $wf^{(s+1)} \in L_\infty[0,1]$ too.

Proposition 3.20. *Let $s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (11). Set $s'' := \max\{3, s\}$. If $0 < \gamma_0, \gamma_1 \le s+1$, then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s+3}(0,1)$ and $wf^{(s'')}, w\varphi^4 f^{(s+4)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\left\|w\left(B_n f - f - \frac{1}{2n}\,Df\right)^{(s)}\right\| \le \frac{c}{n^2}\left(\|wf^{(s'')}\| + \|w\varphi^4 f^{(s+4)}\|\right).$$

If $\gamma_0\gamma_1 = 0$ and still $0 \le \gamma_0, \gamma_1 \le s+1$, then

$$\begin{aligned}\left\|w\left(B_n f - f - \frac{1}{2n}\,Df\right)^{(s)}\right\| \\ \le \frac{c}{n^2}\left(\|wf^{(s'')}\| + \|wf^{(s+2)}\| + \|w\varphi^4 f^{(s+4)}\|\right)\end{aligned}$$

provided that $wf^{(s+2)} \in L_\infty[0,1]$ too.

In addition, we make use of the following Bernstein-type inequalities.

Proposition 3.23. *Let $\ell, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (11) as $0 \le \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there hold:*

(a) $\|w\varphi^{2\ell}(B_nf)^{(2\ell+s)}\| \le c\, n^{\ell}\|wf^{(s)}\|$;

(b) $\|w(B_nf)^{(\ell+s)}\| \le c\, n^{\ell}\|wf^{(s)}\|$.

In order to establish Theorems 3.3 and 3.8, we derive from those estimates the following ones in terms of the differential operator D (to recall, $Df(x) := x(1-x)f''(x)$):

(a) $\|w(B_nf-f)^{(s)}\| \le \dfrac{c}{n}\,\|w(Df)^{(s)}\|, \quad f\in C^{s+2}[0,1]$;

(b) $\left\|w\left(B_nf-f-\dfrac{1}{2n}\,Df\right)^{(s)}\right\| \le \dfrac{c}{n^2}\,\|w(D^2f)^{(s)}\|, \quad f\in C^{s+4}[0,1]$;

(c) $\|w(DB_nf)^{(s)}\| \le cn\|wf^{(s)}\|, \quad f\in C[0,1],\ f\in AC_{loc}^{s-1}(0,1)$, $wf^{(s)}\in L_\infty[0,1]$;

(d) $\|w(D^2B_nf)^{(s)}\| \le c\,n\,\|w(Df)^{(s)}\|, \quad f\in C^{s+2}[0,1]$.

We still assume that $0\le\gamma_0,\gamma_1<s$ for $w:=w(\gamma_0,\gamma_1)$, given in (11). Then Theorems 3.3 and 3.8 follow from [14, Theorems 3.2 and 3.4].

We establish a one-term strong converse inequality for the rate of the weighted simultaneous approximation by B_n for lower order derivatives and additional restrictions on the weight exponents, but still including the case $w=1$.

Theorem 3.26. *Let $s\in\mathbb{N}_+$ as $s\le 6$, and let $w:=w(\gamma_0,\gamma_1)$ be given by (11) with $\gamma_0,\gamma_1\in[0,s/2]$. Then there exists $n_0\in\mathbb{N}_+$ such that for all $f\in C[0,1]$ with $f\in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)}\in L_\infty[0,1]$, and all $n\in\mathbb{N}_+$ with $n\ge n_0$ there holds*

$$K_s^D(f^{(s)},n^{-1})_w \le c\,\|w(B_nf-f)^{(s)}\|.$$

To prove this converse inequality, we again apply the method developed by Ditzian and Ivanov [14], as we establish improvements of Proposition 3.14 and 3.23, which show that the more iterates we take of B_n the smoother its image becomes. More precisely, we prove that if $1\le s\le 6$, $m\ge 2$ and $w:=w(\gamma_0,\gamma_1)$ is given by (11) with $\gamma_0,\gamma_1\in[0,s/2]$, then for all $f\in C^{s+2}[0,1]$ and $n\in\mathbb{N}_+$ such that $n\ge m+s+2$ there holds

$$\|w(D^2B_n^mf)^{(s)}\| \le c'\sqrt{\frac{\log m}{m}}\,n\|w(Df)^{(s)}\|,$$

where the constant c' is independent of f, n and m.

Theorem 3.26 holds for $s = 0$ as well (see [40, 53]). Its assertion for $s = 1$ and $w = 1$ has already been established in [34].

Combining Theorems 3.3 and 3.26, we verify that the error of the weighted simultaneous approximation by the Bernstein operator is equivalent to the K-functional $K_s^D(f^{(s)}, n^{-1})_w$. Thus the following characterization of the rate of the weighted simultaneous approximation by the Bernstein operator holds true.

Theorem 3.30. *Let $s \in \mathbb{N}_+$ as $s \le 6$, and let $w := w(\gamma_0, \gamma_1)$ be given by* (11) *with $\gamma_0, \gamma_1 \in [0, s/2]$. Then there exists $n_0 \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ with $n \ge n_0$ there holds*

$$\|w(B_n f - f)^{(s)}\| \sim K_s^D(f^{(s)}, n^{-1})_w.$$

Similarly, Theorems 3.5 and 3.30 along with (13)-(14) yield

Theorem 3.31. *Let $s \in \mathbb{N}_+$, as $s \le 6$, and $w := w(\gamma_0, \gamma_1)$ be given by* (11). *Then there exists $n_0 \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ with $n \ge n_0$ there hold:*

$$\|w(B_n f - f)'\| \sim \omega_\varphi^2(f', n^{-1/2})_w + \omega_1(f', n^{-1})_w, \quad s = 1,\ 0 \le \gamma_0, \gamma_1 \le 1/2,$$

$$\|(B_n f - f)^{(s)}\| \sim \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + n^{-1}\|f^{(s)}\|, \qquad 2 \le s \le 6,\ \gamma_0 = \gamma_1 = 0,$$

$$\|w(B_n f - f)^{(s)}\| \sim \omega_\varphi^2(f^{(s)}, n^{-1/2})_w + n^{-1}\|wf^{(s)}\|, \qquad 2 \le s \le 6,\ 0 < \gamma_0, \gamma_1 \le s/2.$$

To compare, the characterization in the case $s = 0$ is of the form (see (9))

$$\|B_n f - f\| \sim \omega_\varphi^2(f, n^{-1/2}).$$

Results about the simultaneous approximation by the Bernstein operator can be easily transferred to the Kantorovich operator. The Kantorovich operators or polynomials are defined for $f \in L[0,1]$ and $x \in [0,1]$ by

$$K_n f(x) := \sum_{k=0}^{n} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt\, p_{n,k}(x),\ p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

They are related to the Bernstein polynomials as follows

$$K_n f(x) = (B_{n+1}F(x))', \quad F(x) := \int_0^x f(t)\,dt. \tag{18}$$

More generally, we set for $f \in L[0,1]$ and $m \in \mathbb{N}_+$ (see [50])

$$K_n^{\langle m \rangle} f(x) := (B_{n+m}F_m(x))^{(m)}, \tag{19}$$

where

$$F_m(x) := \frac{1}{(m-1)!}\int_0^x (x-t)^{m-1} f(t)\,dt.$$

The operator $K_n^{\langle m \rangle}$ is referred to as the generalized Kantorovich operator of order m. That generalization of the Kantorovich polynomials or similar modifications of related operators were studied in [9, 10, 29, 30, 32, 35].

All the above results about simultaneous approximation by B_n can be transferred to $K_n^{\langle m \rangle}$. In particular, we have the following characterization of the rate of the simultaneous approximation by $K_n^{\langle m \rangle}$.

Theorem 3.41 *Let $m \in \mathbb{N}_+$, $s \in \mathbb{N}_0$ and $w := w(\gamma_0, \gamma_1)$ be given by (11) as $0 \le \gamma_0, \gamma_1 < s+m$. Then for all $f \in L_\infty[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(K_n^{\langle m \rangle} f - f)^{(s)}\| \le c\, K_{s+m}^D(f^{(s)}, n^{-1})_w.$$

Conversely, there exists $R \in \mathbb{N}_+$ such that for all $f \in L_\infty[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $\ell, n \in \mathbb{N}_+$ with $\ell \ge Rn$ there holds

$$K_{s+m}^D(f^{(s)}, n^{-1})_w \le c \left(\frac{\ell}{n}\right)^r \left(\|w(K_n^{\langle m \rangle} f - f)^{(s)}\| + \|w(K_\ell^{\langle m \rangle} f - f)^{(s)}\|\right).$$

In particular,

$$K_{s+m}^D(f^{(s)}, n^{-r})_w \le c \left(\|w(K_n^{\langle m \rangle} f - f)^{(s)}\| + \|w(K_{Rn}^{\langle m \rangle} f - f)^{(s)}\|\right).$$

The value of the constant c is independent of f, n and ℓ.

In the statement of the last theorem, the condition $f \in AC_{loc}^{s-1}(0,1)$ is to be ignored for $s = 0$.

By virtue of Theorem 3.30, we have the following characterization of the rate of approximation of the Kantorovich operator.

Theorem 3.44. *Let $w := w(\gamma_0, \gamma_1)$ be given by* (11) *with $\gamma_0, \gamma_1 \in [0, 1/2]$. Then there exists $n_0 \in \mathbb{N}_+$ such that for all $f \in L[0,1]$ with $wf \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ with $n \ge n_0$ there holds*

$$\|w(K_n f - f)\| \sim K_1^D(f, n^{-1})_w.$$

The direct estimate for the Kantorovich operator in the case $w = 1$ and $s = 0$ is due to Berens and Xu [3, Theorem 6]. There a weak converse inequality was established as well. The corresponding one-term strong converse inequality and the characterization of the K-functional by the Ditzian-Totik modulus were proved by Gonska and Zhou [34]. Mache [45] established the direct estimate for the Kantorovich operator and a weak converse one in the case $w = \varphi^{2\ell}$ and $s = 2\ell$, $\ell \in \mathbb{N}_+$. All those results were obtained in the L_p-norm, $1 \le p \le \infty$.

Weighted simultaneous approximation by iterated Boolean sums of Bernstein operators

One way to increase the approximation rate of the Bernstein operator B_n is to form its iterated Boolean sums $\mathcal{B}_{r,n} : C[0,1] \to C[0,1]$, defined by

$$\mathcal{B}_{r,n} := I - (I - B_n)^r,$$

where I stands for the identity and $r \in \mathbb{N}_+$. In [46] it was shown that their saturation order is n^{-r}.

An important and nice characterization of the error of $\mathcal{B}_{r,n}$ was given by Gonska ans Zhou [33]. They established the following upper estimate

$$\|\mathcal{B}_{r,n} f - f\| \le c\left(\omega_\varphi^{2r}(f, n^{-1/2}) + \frac{1}{n^r}\|f\|\right), \quad f \in C[0,1],\ n \in \mathbb{N}_+. \tag{20}$$

A Stechkin-type converse inequality was also proved. That enabled them to deduce the trivial class of the operator and a big O equivalence characterization of the error.

Since B_n preserve the algebraic polynomials of degree at most 1, replacing in (20) f with $f - p_1$, where p_1 is the polynomial of degree 1 of best approximation of f in the uniform norm on $[0,1]$, we immediately arrive at

$$\|\mathcal{B}_{r,n} f - f\| \le c\left(\omega_\varphi^{2r}(f, n^{-1/2}) + \frac{1}{n^r}E_1(f)\right), \quad f \in C[0,1],\ n \in \mathbb{N}_+, \tag{21}$$

where $E_1(f)$ denotes the best approximation of f by algebraic polynomials of degree 1 in the uniform norm on $[0,1]$.

Later on Ding and Cao [13] characterized the error of the multivariate generalization of $\mathcal{B}_{r,n}$ on the simplex. In the univariate case, the direct inequality they proved is of the form

$$\|\mathcal{B}_{r,n}f - f\| \le c\, K^D_{r,0}(f, n^{-r}), \quad f \in C[0,1],\ n \in \mathbb{N}_+, \tag{22}$$

where

$$K^D_{r,0}(f,t) := \inf_{g \in C^{2r}[0,1]} \{\|f-g\| + t\|D^r g\|\}$$

with, to recall, $Dg := \varphi^2 g''$ and $\varphi(x) := \sqrt{x(1-x)}$.

They also proved a strong converse inequality of type D (in the terminology introduced in [14]), that is

$$K^D_{r,0}(f, n^{-r}) \le c \max_{k \ge n} \|\mathcal{B}_{r,k}f - f\|, \quad f \in C[0,1],\ n \in \mathbb{N}_+.$$

However, as we show,

$$K^D_{r,0}(f,t) \sim K_{2r,\varphi}(f,t) + tE_1(f), \quad 0 < t \le 1.$$

Therefore, taking also into account (17), we see that the function characteristics on the right side of (21) and (22) are equivalent.

In addition, we establish that

$$K^D_{r,0}(f, n^{-r}) \sim \omega^{2r}_\varphi(f, n^{-1/2}) + \omega^2_\varphi(f, n^{-r/2}), \quad f \in C[0,1],\ n \ge r^2.$$

When we apply it in (22), we get the direct estimate

$$\|\mathcal{B}_{r,n}f - f\| \le c\left(\omega^{2r}_\varphi(f, n^{-1/2}) + \omega^2_\varphi(f, n^{-r/2})\right), \quad f \in C[0,1],\ n \ge r^2.$$

We demonstrate that results on the simultaneous approximation by B_n easily yield (20). In addition, we prove the following strong converse inequality, which improves the earlier converse estimates.

Theorem 4.2. *Let $r \in \mathbb{N}_+$. Then there exists $R \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ and $k, n \in \mathbb{N}_+$ with $k \ge Rn$ there holds*

$$K^D_{r,0}(f, n^{-r}) \le c \left(\frac{k}{n}\right)^r (\|\mathcal{B}_{r,n}f - f\| + \|\mathcal{B}_{r,k}f - f\|).$$

In particular,

$$K^D_{r,0}(f,n^{-r}) \le c\left(\|\mathcal{B}_{r,n}f - f\| + \|\mathcal{B}_{r,Rn}f - f\|\right).$$

The value of the constant c is independent of f, n and k.

In order to prove this theorem we apply [14, Theorem 3.2]. To this end, we verify the following Voronovskaya- and Bernstein-type inequalities:

(a) $\left\|\mathcal{B}_{r,n}g - g - \dfrac{(-1)^{r-1}}{(2n)^r}\, D^r g\right\| \le \dfrac{c}{n^{r+1}}\,\|D^{r+1}g\|, \quad g \in C^{2r+2}[0,1];$

(b) $\|D^r\mathcal{B}_{r,n}f\| \le c\, n^r\|f\|, \quad f \in C[0,1];$

(c) $\|D^{r+1}\mathcal{B}_{r,n}g\| \le c\, n\|D^r g\|, \quad g \in C^{2r}[0,1].$

We characterize the error of the weighted simultaneous approximation by $\mathcal{B}_{r,n}$ by means of the K-functional

$$K^D_{r,s}(f,t)_w := \inf_{g\in C^{2r+s}[0,1]}\left\{\|w(f-g^{(s)})\| + t\|w(D^r g)^{(s)}\|\right\}.$$

We establish the following direct estimate.

Theorem 4.3. *Let $r,s \in \mathbb{N}_+$ and $w := w(\gamma_0,\gamma_1)$ be given by (11) as $0 \le \gamma_0,\gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC^{s-1}_{loc}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \le c\, K^D_{r,s}(f^{(s)}, n^{-r})_w.$$

This estimate can be simplified. We characterize the involved K-functional $K^D_{r,s}(f,t)_w$ by the simpler ones $K_{2r,\varphi}(f,t)_w$ and $K_m(f,t)_w$.

Theorem 4.4. *Let $r,s \in \mathbb{N}_+$ and $w := w(\gamma_0,\gamma_1)$ be given by (11) with $0 < \gamma_0,\gamma_1 < s$. Then for all $wf \in L_\infty[0,1]$ and $0 < t \le 1$ there holds*

$$K^D_{r,s}(f,t)_w \sim \begin{cases} K_{2r,\varphi}(f,t)_w + K_1(f,t)_w, & s = 1,\\[2mm] K_{2r,\varphi}(f,t)_w + t\,\|wf\|, & s \ge 2.\end{cases}$$

The result in the case $w = 1$ is of a different form.

Theorem 4.5. *Let $r,s \in \mathbb{N}_+$. Then for all $f \in C[0,1]$ and $0 < t \le 1$ there holds*

$$K^D_{r,s}(f,t)_1 \sim \begin{cases} K_{2r,\varphi}(f,t) + K_r(f,t) + K_1(f,t), & s = 1,\\[2mm] K_{2r,\varphi}(f,t) + K_r(f,t) + t\,\|f\|, & s \ge 2.\end{cases}$$

Further, by virtue of (16) and (17), we get the following Jackson-type estimates.

Theorem 4.7. *Let $r, s \in \mathbb{N}_+$ and $w = w(\gamma_0, \gamma_1)$ be given by (11) as $0 < \gamma_0, \gamma_1 < s$. Then for all $f \in C[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \le c \begin{cases} \omega_\varphi^{2r}(f', n^{-1/2})_w + \omega_1(f', n^{-r})_w, & s = 1, \\ \omega_\varphi^{2r}(f^{(s)}, n^{-1/2})_w + \dfrac{1}{n^r}\,\|wf^{(s)}\|, & s \ge 2. \end{cases}$$

Theorem 4.8. *Let $r, s \in \mathbb{N}_+$. Then for all $f \in C^s[0,1]$ and $n \in \mathbb{N}_+$ there holds*

$$\|(\mathcal{B}_{r,n}f - f)^{(s)}\| \le c \begin{cases} \omega_\varphi^{2r}(f', n^{-1/2}) + \omega_r(f', n^{-1}) + \omega_1(f', n^{-r}), & s = 1, \\ \omega_\varphi^{2r}(f^{(s)}, n^{-1/2}) + \omega_r(f^{(s)}, n^{-1}) + \dfrac{1}{n^r}\,\|f^{(s)}\|, & s \ge 2. \end{cases}$$

The direct estimates above are sharp. We verify a strong converse inequality that matches the direct one in Theorem 4.3.

Theorem 4.10. *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (11) as $0 \le \gamma_0, \gamma_1 < s$. Then there exists $R \in \mathbb{N}_+$ such that for all $f \in C[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $k, n \in \mathbb{N}_+$ with $k \ge Rn$ there holds*

$$K_{r,s}(f^{(s)}, n^{-r})_w \le c \left(\frac{k}{n}\right)^r \left(\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| + \|w(\mathcal{B}_{r,k}f - f)^{(s)}\|\right).$$

In particular,

$$K_{r,s}(f^{(s)}, n^{-r})_w \le c \left(\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| + \|w(\mathcal{B}_{r,Rn}f - f)^{(s)}\|\right).$$

The value of the constant c is independent of f, n and k.

The proof of Theorems 4.3 and 4.10 is based on the extension of Propositions 3.14, 3.17, 3.20 and 3.23 to $\mathcal{B}_{r,n}$. This extension yields:

(a) $\|w(\mathcal{B}_{r,n}f - f)^{(s)}\| \le \dfrac{c}{n^r}\,\|w(D^r f)^{(s)}\|, \quad f \in C^{2r+s}[0,1];$

(b) $\left\|w\left(\mathcal{B}_{r,n}f - f - \dfrac{(-1)^{r-1}}{(2n)^r}\,D^r f\right)^{(s)}\right\| \le \dfrac{c}{n^{r+1}}\,\|w(D^{r+1}f)^{(s)}\|,$
$f \in C^{2r+s+2}[0,1];$

(c) $\|w(D^r\mathcal{B}_{r,n}f)^{(s)}\| \le c\,n^r\|wf^{(s)}\|, \quad f \in C[0,1],\ f \in AC_{loc}^{s-1}(0,1)$, $wf^{(s)} \in L_\infty[0,1]$;

(d) $\|w(D^{r+1}\mathcal{B}_{r,n}f)^{(s)}\| \le c\,n\,\|w(D^rf)^{(s)}\|, \quad f \in C^{2r+s}[0,1]$.

We still assume that $0 \le \gamma_0, \gamma_1 < s$ for $w := w(\gamma_0,\gamma_1)$, given in (11). Then Theorems 4.3 and 4.10 follow from [14, Theorems 3.2 and 3.4].

Analogously to the simultaneous approximation by the Kantorovich operator, we derive from Theorems 4.3 and 4.10 the following result for the iterated Boolean sums of $K_n^{\langle m\rangle}$ of (19)

$$\mathcal{K}_{r,n}^{\langle m\rangle} := I - (I - K_n^{\langle m\rangle})^r.$$

Theorem 4.25 *Let $m, r \in \mathbb{N}_+$, $s \in \mathbb{N}_0$ and $w := w(\gamma_0,\gamma_1)$ be given by* (11) *as $0 \le \gamma_0, \gamma_1 < s+m$. Then for all $f \in L_\infty[0,1]$ such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $n \in \mathbb{N}_+$ there holds*

$$\|w(\mathcal{K}_{r,n}^{\langle m\rangle}f - f)^{(s)}\| \le c\,K_{r,s+m}^D(f^{(s)}, n^{-r})_w.$$

Conversely, there exists $R \in \mathbb{N}_+$ such that for all $f \in L[0,1]$ with $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_\infty[0,1]$, and all $\ell, n \in \mathbb{N}_+$ with $\ell \ge Rn$ there holds

$$K_{r,s+m}^D(f^{(s)}, n^{-r})_w \le c\left(\frac{k}{n}\right)^r \left(\|w(\mathcal{K}_{r,n}^{\langle m\rangle}f - f)^{(s)}\| + \|w(\mathcal{K}_{r,\ell}^{\langle m\rangle}f - f)^{(s)}\|\right).$$

In particular,

$$K_{r,s+m}^D(f^{(s)}, n^{-r})_w \le c\left(\|w(\mathcal{K}_{r,n}^{\langle m\rangle}f - f)^{(s)}\| + \|w(\mathcal{K}_{r,Rn}^{\langle m\rangle}f - f)^{(s)}\|\right).$$

The value of the constant c is independent of f, n and ℓ.

In the statement of the last theorem the condition $f \in AC_{loc}^{s-1}(0,1)$ is to be ignored for $s = 0$.

Simultaneous approximation by Bernstein polynomials with integer coefficients

Bernstein [5] posed the problem of determining to what extent the requirement on the coefficients of the algebraic polynomials to be integers affects the order of the best algebraic approximation in the uniform norm. To solve this problem Kantorovich [39] (or e.g. [44, Chapter 2, Theorem 4.1]) introduced an integer modification of B_n. It is given by

$$\widetilde{B}_n(f)(x) := \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] x^k (1-x)^{n-k}.$$

Above $[\alpha]$ denotes the largest integer that is less than or equal to the real α. L. Kantorovich showed that if $f \in C[0,1]$ is such that $f(0), f(1) \in \mathbb{Z}$, then

$$\lim_{n\to\infty} \|\widetilde{B}_n(f) - f\| = 0.$$

Clearly, the conditions $f(0), f(1) \in \mathbb{Z}$ are also necessary in order to have $\lim_{n\to\infty} \widetilde{B}_n(f)(0) = f(0)$ and $\lim_{n\to\infty} \widetilde{B}_n(f)(1) = f(1)$, respectively.

Following L. Kantorovich and applying (4), we get a direct estimate of the error of $\widetilde{B}_n$ for $f \in C[0,1]$ with $f(0), f(1) \in \mathbb{Z}$. For $x \in [0,1]$ and $n \in \mathbb{N}_+$ we have

$$|\widetilde{B}_n(f)(x) - f(x)| \le c\,\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}. \tag{23}$$

We show that the simultaneous approximation by $\widetilde{B}_n(f)$ satisfies a similar estimate. Before stating that result, let us note that another integer modification of $B_n f$ possesses actually better properties regarding simultaneous approximation. In it, instead of the integer part $[\alpha]$ we use the nearest integer $\langle\alpha\rangle$ to the real α. More precisely, if $\alpha \in \mathbb{R}$ is not the arithmetic mean of two consecutive integers, we set $\langle\alpha\rangle$ to be the integer at which the minimum $\min_{m\in\mathbb{Z}} |\alpha - m|$ is attained. When α is right in the middle between two consecutive integers, we can define $\langle\alpha\rangle$ to be either of them even without following a given rule. The results we prove are valid regardless of our choice in this case.

We denote that integer modification of the Bernstein polynomial by $\widehat{B}_n(f)$, that is, we set

$$\widehat{B}_n(f)(x) := \sum_{k=0}^{n} \left\langle f\left(\frac{k}{n}\right) \binom{n}{k} \right\rangle x^k (1-x)^{n-k}$$

for $f \in C[0,1]$ and $x \in [0,1]$.

Similarly to (23), we have

$$\|\widehat{B}_n(f) - f\| \le c\,\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{2n} \tag{24}$$

for all $f \in C[0,1]$ with $f(0), f(1) \in \mathbb{Z}$ and all $n \in \mathbb{N}_+$.

Combining (23) and (24) with (9), we arrive at the characterization

$$\begin{aligned} c^{-1}\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right) &\le \|\widetilde{B}_n(f) - f\| + \frac{1}{n} \\ &\le c\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right) \end{aligned}$$

and

$$\begin{aligned} c^{-1}\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right) &\le \|\widehat{B}_n(f) - f\| + \frac{1}{n} \\ &\le c\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right) \end{aligned}$$

valid for all $f \in C[0,1]$ with $f(0), f(1) \in \mathbb{Z}$ and all $n \ge n_0$ with some n_0, which is independent of f.

Consequently, if $0 < \alpha \le 1$, then

$$\|\widetilde{B}_n(f) - f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad \omega_\varphi^2(f, h) = O(h^{2\alpha}) \tag{25}$$

and

$$\|\widehat{B}_n(f) - f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad \omega_\varphi^2(f, h) = O(h^{2\alpha}), \tag{26}$$

provided that $f(0), f(1) \in \mathbb{Z}$; we assume $f \in C[0,1]$.

In addition, we prove that the approximation processes generated by $\widetilde{B}_n$ and $\widehat{B}_n$ in the uniform norm on $[0,1]$ are saturated with the saturation rate of $1/n$ and if $\|\widetilde{B}_n(f) - f\| = o(1/n)$ or $\|\widehat{B}_n(f) - f\| = o(1/n)$, then, similarly to the Bernstein operator, we have that $\widetilde{B}_n(f) = \widehat{B}_n(f) = f$ and f is a polynomial of the type $px + q$, where $p, q \in \mathbb{Z}$. As it follows from (25)-(26), their saturation class consists of those functions $f \in AC[0,1]$ such that $f(0), f(1) \in \mathbb{Z}$, $f' \in AC_{loc}(0,1)$ and $\varphi^2 f'' \in L_\infty[0,1]$.

Let us explicitly note that for any fixed $n \ge 2$ the operator $\widetilde{B}_n : C[0,1] \to C[0,1]$ is not bounded in the sense that there does *not* exist a constant M such that

$$\|\widetilde{B}_n f\| \le M\,\|f\| \quad \forall\, f \in C[0,1].$$

Therefore we cannot drop the quantity $1/n$ on the right-hand side of the estimate (23), or replace it with $c\,\|f\|\,n^{-1}$. That operator is not continuous either. On the other hand, $\widehat{B}_n$ is bounded but not continuous. Both operators are not linear. To emphasize the latter we write $\widetilde{B}_n(f)$ and $\widehat{B}_n(f)$, not $\widetilde{B}_n f$ and $\widehat{B}_n f$.

We verify that the integer forms of the Bernstein polynomials $\widetilde{B}_n$ and $\widehat{B}_n$ possess the property of simultaneous approximation and establish a direct estimate of the rate of approximation.

Theorem 5.1. *Let $s \in \mathbb{N}_+$. Let $f \in C^s[0,1]$ be such that*

$$f(0), f(1), f'(0), f'(1) \in \mathbb{Z} \text{ and } f^{(i)}(0) = f^{(i)}(1) = 0,\ i = 2, \dots, s.$$

Let also there exist $n_0 \in \mathbb{N}_+$, $n_0 \ge s$, such that

$$\begin{aligned} f\left(\frac{k}{n}\right) &\ge f(0) + \frac{k}{n} f'(0), \quad k = 1, \dots, s,\ n \ge n_0, \\ f\left(\frac{k}{n}\right) &\ge f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s, \dots, n-1,\ n \ge n_0. \end{aligned}$$

Then for $n \ge n_0$ there holds

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| \le c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}) + \dfrac{1}{n}, & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\| + \dfrac{1}{n}, & s \ge 2. \end{cases}$$

The value of the constant c is independent of f and n.

Remark 5.3. An analogous result holds for the integer form of the Bernstein operator defined by means of the ceiling function instead of the integer part. Then we assume that the reverse inequalities for $f(k/n)$ hold, that is,

$$\begin{aligned} f\left(\frac{k}{n}\right) &\le f(0) + \frac{k}{n} f'(0), \quad k = 1, \dots, s,\ n \ge n_0, \\ f\left(\frac{k}{n}\right) &\le f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s, \dots, n-1,\ n \ge n_0. \end{aligned}$$

The estimates of the rate of convergence for $\widehat{B}_n$ are valid under *weaker* assumptions.

Theorem 5.4. *Let $s \in \mathbb{N}_+$. Let $f \in C^s[0,1]$ be such that*

$$f(0), f(1), f'(0), f'(1) \in \mathbb{Z} \text{ and } f^{(i)}(0) = f^{(i)}(1) = 0,\ i = 2, \dots, s.$$

Then

$$\|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| \\ \le c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}) + \dfrac{1}{n}, & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\| + \dfrac{1}{n}, & s \ge 2. \end{cases}$$

The value of the constant c is independent of f and n.

In addition, we show that the assumptions made in Theorems 5.1 and 5.4 are necessary in order to have uniform simultaneous approximation. Concerning the difference between the set of conditions on the derivatives for $s = 1$ and $s \ge 2$, let us note that $\widetilde{B}_n$ and $\widehat{B}_n$ preserve the polynomials of the form $p\,x + q$, where $p, q \in \mathbb{Z}$ (that is verified just as for the Bernstein operator). Therefore it is not surprising that there are not any restrictions on the values of the function and its first derivative at the endpoints except that they must be integers. However, the requirement that the derivatives of order 2 and higher must be equal to 0 at the endpoints is quite unexpected. Technically, it is related to the fact that $\left(\frac{k}{n}\right)^s \binom{n}{k} \in \mathbb{Z}$ for all k and n iff $s = 0$ or $s = 1$.

We establish the following weak converse relations that complement the direct estimates in Theorems 5.1 and 5.4.

Theorem 5.5. *Let $s \in \mathbb{N}_+$ and $0 < \alpha < 1$. Let $f \in C^s[0,1]$, $f(0), f(1) \in \mathbb{Z}$, and*

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha}) \quad \text{or} \quad \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha}).$$

Then

$$\omega_\varphi^2(f^{(s)}, h) = O(h^{2\alpha}) \quad \text{and} \quad \omega_1(f^{(s)}, h) = O(h^\alpha).$$

The proof is based on application of the Berens-Lorentz Lemma (see [2], or e.g. [12, Chapter 10, Lemma 5.2])

Combining this theorem with Theorems 5.1 and 5.4, we get the following two big O equivalence relations.

Corollary 5.6. *Let $s \in \mathbb{N}_+$ and $0 < \alpha < 1$. Let $f \in C^s[0,1]$ be such that $f(0), f(1), f'(0), f'(1) \in \mathbb{Z}$ and $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2, \dots, s$. Let also there exist $n_0 \in \mathbb{N}_+$, $n_0 \geq s$, such that*

$$\begin{aligned} f\left(\frac{k}{n}\right) &\geq f(0) + \frac{k}{n} f'(0), \quad k = 1, \dots, s, \ n \geq n_0, \\ f\left(\frac{k}{n}\right) &\geq f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s, \dots, n-1, \ n \geq n_0. \end{aligned}$$

Then

$$\begin{aligned} &\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha}) \\ &\qquad \iff \quad \omega_\varphi^2(f^{(s)}, h) = O(h^{2\alpha}) \quad \text{and} \quad \omega_1(f^{(s)}, h) = O(h^\alpha). \end{aligned}$$

Corollary 5.7. *Let $s \in \mathbb{N}_+$ and $0 < \alpha < 1$. Let $f \in C^s[0,1]$ be such that $f(0), f(1), f'(0), f'(1) \in \mathbb{Z}$ and $f^{(i)}(0) = f^{(i)}(1) = 0$, $i = 2, \dots, s$. Then*

$$\begin{aligned} &\|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha}) \\ &\qquad \iff \quad \omega_\varphi^2(f^{(s)}, h) = O(h^{2\alpha}) \quad \text{and} \quad \omega_1(f^{(s)}, h) = O(h^\alpha). \end{aligned}$$

The proof of the results above is based on the relation between $(B_n f)^{(s)}$ and $(\widetilde{B}_n(f))^{(s)}$, given by

$$\|(B_n f)^{(s)} - (\widetilde{B}_n(f))^{(s)}\| \leq c \left(\omega_1(f^{(s)}, n^{-1}) + \frac{1}{n}\right)$$

and similarly between $(B_n f)^{(s)}$ and $(\widehat{B}_n(f))^{(s)}$ under the assumptions of Theorems 5.1 and 5.4, respectively.

Following the relation between the Kantorovich polynomials and the Bernstein polynomials given in (18), we define

$$\widehat{K}_n(f)(x) := \left(\widehat{B}_{n+1}(F)(x)\right)', \quad F(x) := \int_0^x f(t)\, dt,$$

where $f \in L[0,1]$ and $x \in [0,1]$.

Then we have

$$\begin{aligned} \widehat{K}_n(f)(x) = \sum_{k=0}^{n} \Bigg(&(k+1) \left\langle \int_0^{\frac{k+1}{n+1}} f(t)\, dt \binom{n+1}{k+1} \right\rangle \\ &- (n-k+1) \left\langle \int_0^{\frac{k}{n+1}} f(t)\, dt \binom{n+1}{k} \right\rangle \Bigg) x^k (1-x)^{n-k}. \end{aligned}$$

Now, Theorem 5.4 implies the following direct estimate of the rate of simultaneous approximation by $\widehat{K}_n$.

Theorem 5.17 *Let $s \in \mathbb{N}_0$. Let $f \in C^s[0,1]$ be such that*

$$\int_0^1 f(t)\,dt \in \mathbb{Z}, \quad f(0), f(1) \in \mathbb{Z},$$
$$f^{(i)}(0) = f^{(i)}(1) = 0,\ i = 1, \dots, s.$$

Then

$$\|(\widehat{K}_n(f))^{(s)} - f^{(s)}\| \le c \begin{cases} \omega_\varphi^2(f, n^{-1/2}) + \omega_1(f, n^{-1}) + \dfrac{1}{n}, & s = 0, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\,\|f^{(s)}\| + \dfrac{1}{n}, & s \ge 1. \end{cases}$$

The value of the constant c is independent of f and n.

Clearly, the only advantage of $\widehat{K}_n$ to $\widehat{B}_n$ could be that it is defined by integrals of f rather than its values, which is useful in case the former are more readily available than the latter.

Direct and converse Voronovskaya estimates for the Bernstein operator

We estimate the rate of the convergence in the Voronovskaya's theorem [54], which states that if $f \in C^2[0,1]$, then

$$\lim_{n\to\infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2}\, f''(x)$$

uniformly on $[0,1]$.

We introduce the linear operator

$$D_n f(x) := n(B_n f(x) - f(x))$$

and refer to it as the Voronovskaya operator.

We consider it on the Sobolev-type function spaces

$$W_\infty^m(\varphi)[0,1] := \{f \in C[0,1] : f \in AC_{loc}^{m-1}(0,1), \varphi^m f^{(m)} \in L_\infty[0,1]\},$$

where, to recall, $\varphi(x) := \sqrt{x(1-x)}$. We have $W_\infty^{m+1}(\varphi)[0,1] \subset W_\infty^m(\varphi)[0,1]$.

For $f \in W_\infty^2(\varphi)[0,1]$ we set $\mathcal{D}f(x) := \frac{\varphi^2(x)}{2} f''(x)$.

It is known that (see [14, Lemma 8.3])

$$\left\| B_n f - f - \frac{1}{2n}\, \varphi^2 f'' \right\| \le \frac{c}{n^{3/2}} \|\varphi^3 f^{(3)}\|, \quad f \in W_\infty^3(\varphi)[0,1],$$

which can be written in the form

$$\|D_n f - \mathcal{D}f\| \le \frac{c}{n^{1/2}} \|\varphi^3 f^{(3)}\|, \quad f \in W_\infty^3(\varphi)[0,1].$$

We show, assuming a higher degree of smoothness, that

$$\left\| B_n f - f - \frac{1}{2n}\, \varphi^2 f'' \right\| \le \frac{c}{n^2} \left(\|\varphi^2 f^{(3)}\| + \|\varphi^4 f^{(4)}\| \right), \quad f \in W_\infty^4(\varphi)[0,1],$$

that is,

$$\|D_n f - \mathcal{D}f\| \le \frac{c}{n} \left(\|\varphi^2 f^{(3)}\| + \|\varphi^4 f^{(4)}\| \right).$$

That slightly improves the estimate

$$\left\| B_n f - f - \frac{1}{2n}\, \varphi^2 f'' \right\| \le \frac{c}{n^2} \left(\|f^{(3)}\| + \|f^{(4)}\| \right), \quad f \in C^4[0,1],$$

established in [31].

To state our main results about the convergence rate of D_n we use the K-functionals $K_{2,\varphi}(F,t)_w$ defined in (12) and

$$\widetilde{K}(F,t) := \inf_{g \in W_\infty^4(\varphi)[0,1]} \left\{ \|F - \mathcal{D}g\| + t \left(\|\varphi^2 g^{(3)}\| + \|\varphi^4 g^{(4)}\| \right) \right\}.$$

We establish the following characterization of the rate of approximation of $\mathcal{D}f$ by means of $D_n f$.

Theorem 6.1. *For all $f \in W_\infty^2(\varphi)[0,1]$ and all $n \in \mathbb{N}_+$ there holds*

$$\|D_n f - \mathcal{D}f\| \le c\, \widetilde{K}(\mathcal{D}f, n^{-1}) \le c \left(K_{2,\varphi}(f'', n^{-1})_{\varphi^2} + \frac{1}{n} \|\varphi^2 f''\| \right). \tag{27}$$

Conversely, for all $f \in W_\infty^2(\varphi)[0,1]$ and all $k, n \in \mathbb{N}_+$ there holds

$$K_{2,\varphi}(f'', n^{-1})_{\varphi^2} \le 2\, \|D_k f - \mathcal{D}f\| + c\frac{k}{n}\, K_{2,\varphi}(f'', k^{-1})_{\varphi^2} + \frac{c}{n} \|\varphi^2 f''\|. \tag{28}$$

The value of the constant c is independent of f, n and k.

The above two estimates can also be written in the form

$$\left\|B_nf - f - \frac{1}{2n}\,\varphi^2 f''\right\| \le \frac{c}{n}\,\widetilde{K}(\mathcal{D}f, n^{-1}) \le \frac{c}{n}\,K_{2,\varphi}(f'', n^{-1})_{\varphi^2} + \frac{c}{n^2}\,\|\varphi^2 f''\| \tag{29}$$

and

$$\frac{c}{k}\,K_{2,\varphi}(f'', n^{-1})_{\varphi^2} \le 2\left\|B_kf - f - \frac{1}{2k}\,\varphi^2 f''\right\| + \frac{c}{n}\,K_{2,\varphi}(f'', k^{-1})_{\varphi^2} + \frac{c}{nk}\,\|\varphi^2 f''\|. \tag{30}$$

Estimates (27) and (29) can be considered as *direct Voronovskaya inequalities*, and estimates (28) and (30) as *weak converse Voronovskaya inequalities.*

Similar direct point-wise estimates were established in [28, Theorem 3.2] and [52, Theorem 2]. The assumptions on the functions made there are more restrictive. However, the first of these results is very general and both give explicit values to the absolute constant.

We derive the following equivalence relation from Theorem 6.1.

Corollary 6.3. *Let $f \in W^2_\infty(\varphi)[0,1]$ and $0 < \alpha < 1$. Then*

$$\|D_nf - \mathcal{D}f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad K_{2,\varphi}(f'', t)_{\varphi^2} = O(t^\alpha).$$

Bernstein [6] proved that if $f \in C^{2r}[0,1]$, then

$$\lim_{n\to\infty} n^r \left(B_nf(x) - f(x) - \sum_{i=1}^{2r} B_n\left((\circ - x)^i\right)(x)\,\frac{f^{(i)}(x)}{i!}\right) = 0$$

uniformly on $[0,1]$ (see also [51]). A quantitative estimate of this convergence for positive linear operators on $C[0,1]$ was established by Gonska [28].

Setting $r = 2$ above we have for $f \in C^4[0,1]$

$$\lim_{n\to\infty} n(D_nf(x) - \mathcal{D}f(x)) = D'f(x) \tag{31}$$

uniformly on $[0,1]$, where

$$D'f(x) := \frac{(1-2x)\varphi^2(x)}{3!}\,f^{(3)}(x) + \frac{3\,\varphi^4(x)}{4!}\,f^{(4)}(x).$$

This shows that the operator D_n is saturated, as its saturation order is n^{-1} and its trivial class is the set of the algebraic polynomials of degree at most 2.

We establish the following quantitative estimate of the convergence in (31).

Theorem 6.4. *For all $f \in W^4_\infty(\varphi)[0,1]$ and all $n \in \mathbb{N}_+$ there holds*

$$\left\| D_n f - \mathcal{D} f - \frac{1}{n}\, D' f \right\| \le \frac{c}{n}\, K_{2,\varphi^2}(f^{(4)}, n^{-1})_{\varphi^4} + \frac{c}{n^2}\, \|\varphi^4 f^{(4)}\|.$$

The value of the constant c is independent of f and n.

Instead of $K_{2,\varphi}(F,t)_{\varphi^r}$ one can use the weighted Ditzian-Totik modulus of smoothness $\omega^2_\varphi(F,t)_{\varphi^r}$, defined in (15) (see also (17)). In fact, the weighted Ditzian-Totik main-part modulus of smoothness allows us to restate the characterization in Corollary 6.3 in a simpler form.

Corollary 6.5. *Let $f \in W^2_\infty(\varphi)[0,1]$ and $0 < \alpha < 1$. Then*

$$\|D_n f - \mathcal{D} f\| = O(n^{-\alpha}) \quad \Longleftrightarrow \quad \|\varphi^2 \bar{\Delta}^2_{h\varphi} f''\|_{[2h^2, 1-2h^2]} = O(h^{2\alpha}).$$

Embedding inequalities

To establish the results, related to the simultaneous approximation by the Bernstein operator, its iterated Boolean sums and the Voronovskaya operator, we extensively use inequalities between the norms of the derivatives of the functions as well as between them and the norms of the values of the differential operator that is associated with the approximation by the iterated Boolean sums of the Bernstein operator, $(d/dx)^s D^r$, and, in particular, by the Bernstein operator itself.

When we consider the weighted simultaneous approximation by the iterated Boolean sums of B_n, we do not establish the estimates we need directly in terms of the differential operator $(d/dx)^s D^r$, because it is rather involved. Instead, we do so in terms of the norms of the components into which $(D^r g)^{(s)}$ expands. They are of the form $q\varphi^{2i} g^{(j)}$, where q is an algebraic polynomial, which can be ignored, and $i, j \in \mathbb{N}_0$. Then, making use of certain embedding inequalities, we return to $(D^r g)^{(s)}$. That allows us not only to get round the technical difficulties of dealing with $(d/dx)^s D^r$, but also to derive almost simultaneously both characterizations of $\|w(\mathcal{B}_{r,n} f - f)^{(s)}\|$: the more natural one by $K^D_{r,s}(f,t)_w$ and the more useful one by $K_{2r,\varphi}(f,t)_w$ and $K_m(f,t)_w$.

In fact, applying appropriate embedding inequalities is typical for such problems in Approximation Theory; see e.g. [3, Lemmas 2, 3 and 4], [15, p. 135], [33, Lemma 2] and [34, pp. 127-128].

It is well-known that (e.g. [12, Chapter 2, Theorem 5.6])

$$\|f^{(j)}\|_J \le c\left(\|f\|_J + \|f^{(m)}\|_J\right), \quad j = 0, \dots, m,$$

where $f \in W_\infty^m(J)$ and J is an interval on the real line. The value of the constant c is independent of f.

Besides this inequality, we establish and use several more. They are stated in the following two propositions.

Proposition 2.1. *Let $j, m \in \mathbb{N}_0$ as $j < m$. Let $w_\mu := w(\gamma_{\mu,0}, \gamma_{\mu,1})$ be given by (11) with $\gamma_{\mu,0}, \gamma_{\mu,1} > 0$ for $\mu = 1, 2$ and let $\gamma_{2,\nu} \le \gamma_{1,\nu} + m - j$ for $\nu = 0, 1$. Let also $g \in AC_{loc}^{m-1}(0,1)$ be such that $w_2 g^{(m)} \in L_\infty[0,1]$. Then*

$$\|w_1 g^{(j)}\| \le c\left(\|g\|_{[1/4,3/4]} + \|w_2 g^{(m)}\|\right).$$

The value of the constant c is independent of g.

Proposition 2.6 *Let $r, s \in \mathbb{N}_+$ and $w := w(\gamma_0, \gamma_1)$ be given by (11) as $0 \le \gamma_0, \gamma_1 < s$. Set $j_s := 1$ if $s = 1$, and $j_s := 0$ otherwise. Then for all $g \in AC^{2r+s-1}[0,1]$ there hold*

$$\|w g^{(j+s)}\| \le c\,\|w(D^r g)^{(s)}\|, \quad j = j_s, \dots, r,$$

and

$$\|w \varphi^{2r} g^{(2r+s)}\| \le c\,\|w(D^r g)^{(s)}\|.$$

The value of the constant c is independent of g.

Organization of the contents of the dissertation

In Chapter 1 we collect the definitions and the basic properties of the standard K-functionals and moduli of smoothness that are used in problems of the type we consider.

In Chapter 2 we establish inequalities between the weighted essential supremum norms of the derivatives of the functions as well as between them and the norms of the values of the differential operator that is associated with the approximation by the iterated Boolean sums of the Bernstein operator, and, in particular, by the Bernstein operator itself. The results presented in this chapter were published in [17, 18, 24].

In Chapter 3 we establish matching direct and one- or two-term strong converse estimates of the rate of the simultaneous approximation by the Bernstein operator in the weighted essential supremum norm. The material presented in this chapter was published in [18, 19].

In Chapter 4 most of the estimates of the previous chapter are extended to iterated Boolean sums of the Bernstein operator. The results presented in this chapter were published in [16, 17, 18, 22, 23].

In Chapter 5 we establish direct and weak converse estimates for the simultaneous approximation by two modifications of the Bernstein polynomials, which provide approximation by algebraic polynomials with integer coefficients. The results presented in this chapter were published in [20, 21].

In Chapter 6 we characterize the rate of the convergence in the Voronovskaya's theorem. The results presented in this chapter were published in [24], written jointly with I. Gadjev.

Acknowledgments

I am especially grateful to Professor Kamen Ivanov, Professor Dany Leviatan, Kiril Delev and the anonymous Referees for comments and corrections on the manuscripts. I am also indebted to Professor Gancho Tachev for providing me with papers and discussions on problems treated in the dissertation. I especially appreciate the support, encouragement and advice, which I have been receiving from Professor Kamen Ivanov throughout my research. Also, the support I received from Professor Geno Nikolov, Professor Nadejda Ribarska, Associate Professor Ivan Gadjev, Associate Professor Parvan Parvanov and Associate Professor Rumen Uluchev (in alphabetical order) has been very important for me.

References

[1] J. A. Adell, G. Sangüesa, Upper constant in direct inequalities for Bernstein-type operators, *J. Approx. Theory* **109** (2001), 229–241.

[2] H. Berens, G. G. Lorentz, Inverse theorems for Bernstein polynomials, *Indiana Univ. Math. J.* **21** (1972), 693–708.

[3] H. Berens, Y. Xu, On Bernstein-Durrmeyer polynomials with Jacobi weights, In: "Approximation Theory and Functional Analysis" (C. K. Chui, Ed.), 1991, 25–46.

[4] S. N. Bernstein, Démonstration du théorème de Weierstrass, fondeé sur le calcul des probabilités, *Commun. Soc. Math. Kharkow* (2), **13** (1912-13), 1–2.

[5] S. N. Bernstein, Some remarks on polynomials of least deviation with integer coefficients, *Dokl. Akad. Nauk SSSR* **16** (1930), 411–415 (in Russian).

[6] S. N. Bernstein, Complément à l'article de E. Voronovskaya "Détermination de la forme asymptotique de l'approximation des fonctions par les polynômes de M. Bernstein", *Dokl. Acad. Nauk SSSR A* **4** (1932), 86–92.

[7] J. Bustamante, Bernstein Operators and Their Properties, Birkhäuser, Cham, 2017.

[8] P.L. Butzer, R.J. Nessel, Fourier Analysis and Approximation, Birkhäser Verlag, Basel, 1971.

[9] J. de la Cal, A. M. Valle, A generalization of Bernstein-Kantorovič operators, *J. Math. Anal. Appl.* **252** (2000), 750–766.

[10] J.-D. Cao, On Sikkema-Kantorovič polynomials of order k, *Approx. Theory Appl.* **5** (1989), 99–109.

[11] I. Chlodowsky, Erster Mathematikerkongress der USSR, Charkow, 1930.

[12] R. A. DeVore, G. G. Lorentz, Constructive Approximation, Springer-Verlag, Berlin, 1993.

[13] C. Ding, F. Cao, K-functionals and multivariate Bernstein polynomials, *J. Approx. Theory* **155** (2008), 125–135.

[14] Z. Ditzian, K. G. Ivanov, Strong converse inequalities, *J. Anal. Math.* **61** (1993), 61–111.

[15] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer-Verlag, New York, 1987.

[16] B. R. Draganov, Upper estimates of the approximation rate of combinations of iterates of the Bernstein operator, *Annuaire Univ. Sofia Fac. Math. Inform.* **101** (2013), 95–104.

[17] B. R. Draganov, On simultaneous approximation by iterated Boolean sums of Bernstein operators, *Results Math.* **66** (2014), 21–41.

[18] B. R. Draganov, Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums, J*. Approx. Theory* **200** (2015), 92–135.

[19] B. R. Draganov, An exact strong converse inequality for the weighted simultaneous approximation by the Bernstein operator, In: "Constructive Theory of Functions, Sozopol 2016" (K. Ivanov, G. Nikolov, R. Uluchev, Eds.), pp. 75–97, Marin Drinov Academic Publishing House, Sofia, 2018.

[20] B. R. Draganov, Simultaneous approximation by Bernstein polynomials with integer coefficients, *J. Approx. Theory* **237** (2019), 1–16.

[21] B. R. Draganov, Converse estimates for the simultaneous approximation by Bernstein polynomials with integer coefficients, In: "Constructive Theory of Functions, Sozopol 2019" (B. Draganov, K. Ivanov, G. Nikolov, R. Uluchev, Eds.), pp. 39–51, Marin Drinov Academic Publishing House, Sofia, 2020.

[22] B. R. Draganov, Corrigendum to "Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums" [J. Approx. Theory 200 (2015) 92–135], *J. Approx. Theory* **252** (2020), 105321.

[23] B. R. Draganov, A strong converse inequality for the iterated Boolean sums of the Bernstein operator, *Stud. Univ. Babeş-Bolyai Math.* **67** (2022), 591–598.

[24] B. R. Draganov, I. Gadjev, Direct and converse Voronovskaya estimates for the Bernstein operator, *Results Math.* **73**:11 (2018).

[25] B. R. Draganov, K. G. Ivanov, Characterizations of weighted K-functionals and their application, In: "Constructive Theory of Functions, Varna 2005" (B. Bojanov, Ed.), pp. 88-97, Marin Drinov Academic Publishing House, Sofia, 2006.

[26] M. S. Floater, On the convergence of derivatives of Bernstein approximation, *J. Approx. Theory* **134** (2005), 130–135.

[27] I. Gavrea, H. H. Gonska, R. Păltănea, G. Tachev, General estimates for the Ditzian-Totik modulus, *East J. Approx.* **2** (2003), 175–194.

[28] H. H. Gonska, On the degree of approximation in Voronovskaja's theorem, *Stud. Univ. Babeş-Bolyai Math.* **52** (2007), 3, 103–115.

[29] H. Gonska, M. Heilmann, I. Raşa, Asymptotic behaviour of differentiated Bernstein polynomials revisited, *Gen. Math.* **18** (2010), 45–53.

[30] H. Gonska, M. Heilmann, I. Raşa, Kantorovich operators of order k, *Numer. Funct. Anal. Optim.* **32** (2011), 717–738.

[31] H. H. Gonska, I. Raşa, The limiting semigroup of the Bernstein iterates: degree of convergence, *Acta Math. Hung.* **111** (2006), 119–130.

[32] H. Gonska, I. Raşa, Asymptotic behaviour of differentiated Bernstein polynomials, *Mat. Vesnik* **61** (2009), 53–60.

[33] H. Gonska, X.-l. Zhou, Approximation theorems for the iterated Boolean sums of Bernstein operators, *J. Comput. Appl. Math.* **53** (1994), 21–31.

[34] H. Gonska, X.-l. Zhou, The strong converse inequality for Bernstein-Kantorovich operators, *Comput. Math. Appl.* **30** (1995), 103–128.

[35] M. Heilmann, I. Raşa, k-th order Kantorovich type modification of the operators U_n^ρ, *J. Appl. Funct. Anal.* **9** (2014), 320–334.

[36] G. Jiang, On Bernstein operators and its compositions, *J. Zhaoqing Univ.* **26**(5) (2005), 5-7 (in Chinese).

[37] H. B. Jiang, L. S. Xie, Simultaneous approximation by Bernstein operators, *Pure Appl. Math.* (Xi'an) **4**:22 (2006), 471–476 (in Chinese).

[38] H. Johnen, Inequalities connected with moduli of smoothness, *Mat. Vesnik* **3** (1972), 289–305.

[39] L. V. Kantorovich, Some remarks on the approximation of functions by means of polynomials with integer coefficients, *Izv. Akad. Nauk SSSR, Ser. Mat.* **9** (1931), 1163–1168 (in Russian).

[40] H.-B. Knoop, X.-L. Zhou, The lower estimate for linear positive operators (II), *Results Math.* **25** (1994), 315–330.

[41] A. J. López-Moreno, J. Martínez-Moreno, F. J. Muñoz-Delgado, Asymptotic expression of derivatives of Bernstein type operators, *Rend. Circ. Mat. Palermo. Ser. II* **68** (2002), 615–624.

[42] G. G. Lorentz, Zur theorie der polynome von S. Bernstein, *Mat. Sb (N.S.)* **2** (1937), 543–556.

[43] G. G. Lorentz, Bernstein Polynomials, Chelsea Publishing Company, New York, Second Edition, 1986.

[44] G. G. Lorentz, M. v.Golitschek, Y. Makovoz, Constructive Approximation, Advanced Problems, Springer-Verlag, Berlin, 1996.

[45] D. H. Mache, Equivalence theorem on weighted simultaneous L_p-approximation by the method of Kantorovič operators, *J. Approx. Theory* **78** (1994), 321–350.

[46] C. Micchelli, The saturation class and iterates of the Bernstein polynomials, *J. Approx. Theory* **8** (1973), 1–18.

[47] R. Păltănea, Approximation Theory Using Positive Linear Operators, Birkhäuser, Boston, 2004.

[48] T. Popoviciu, Sur l'approximation des fonctions convexes d'ordre supérieur, *Mathematica (Cluj)* **10** (1935), 49–54.

[49] T. Popoviciu, Despre cea mai ba aproximare a functiilor continue prin polinoame, Monografii matematice, Sec. Mat. A Univ. Din Cluj, fasc. III, 1937.

[50] Bl. Sendov, V. A. Popov, Convergence of the derivatives of linear positive operators, *Bulgar. Akad. Nauk Izv. Mat. Inst.* **11** (1970), 107–115 (in Bulgarian).

[51] G. T. Tachev, The complete asymptotic expansion for Bernstein operators, *J. Math. Anal. Appl.* **385** (2012), 1179–1183.

[52] G. Tachev, New estimates in Voronovskaja's theorem, *Numer. Algorithms* **59** (2012), 119–129.

[53] V. Totik, Approximation by Bernstein polynomials, *Amer. J. Math.* **116** (1994), 995–1018.

[54] E. Voronovskaya, Détermnation de la forme asymptotice l'approximation des fonctions par les polynomes de M. Bernstein, *Dokl. Akad. Nauk SSSR* **4** (1932), 79–85.

[55] S. Wigert, Sur l'approximation par polynômes des fonctions continues, *Ark. Math. Astr. Fys.* **22 B** (1932), 1–4.